Lattice representations of Penrose tilings of the plane

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Abstract

Two-, three- and four-dimensional representations of Penrose tilings of the plane are described. The vertices that occur in these representations lie on lattices. Symmetries and methods of visualizing these representations are discussed. The question of efficiently storing the information necessary to reconstruct a tiling is addressed.

I. INTRODUCTION

The subject of Penrose tiles has been studied extensively, and the concept of quasiperiodicity has found applications in physics [1]. The definition of the Penrose tiles discussed in this paper is the same as that of Refs. [1] and [2], but the methods presented here can be applied to other types of tiles as well. We consider a tiling of the plane by a rhombus containing an angle of 36° and a rhombus containing an angle of 72°; the edges of these two tiles have unit length. Some authors work with additional matching rules, according to which certain arrangements of tiles are forbidden. The methods of this paper can be applied when matching rules are present as well as when they are absent. An example of a tiling is shown in Fig. 1. Although there are only finitely many directions possible for the edges in such a tiling, the vertices do not lie on any lattice (a lattice is a set of points obtained by taking all integer linear combinations of a set of linearly independent vectors). The subject of this paper is two-, three- and four-dimensional representations of Penrose tilings of the plane. Given a tiling of the plane, a lattice representation is defined by placing tiles with four vertices on a lattice, which has a dimensionality of two, three or four. The tiles share edges with neighboring tiles in the same way as in the original tiling of the plane. Figure 2 shows an example of a three-dimensional representation of the tiling in Fig. 1.

II. FOUR-DIMENSIONAL REPRESENTATION

We begin by defining a four-dimensional representation of a Penrose tiling of the plane. Such representations are well-known [3–6], but we present a discussion here because the later sections of this paper make use of this material. We define x and y axes in the plane in such a way that one of the vertices of the tiling is located at the origin, and that the angles between edges of tiles and the x axis are multiples of 36°. Because of the relationships

$$\cos 36^\circ = \frac{1+\sqrt{5}}{4}\,,$$
$$\cos 72^\circ = \frac{-1+\sqrt{5}}{4}\,, \tag{2.1}$$

and $\cos 0 = 1$, the x coordinates of all of the vertices in the tiling can be expressed as integer linear combinations of $1/4$ and $\sqrt{5}/4$. The y coordinates of all of the vertices in the tiling can be expressed as integer linear combinations of $\sin 36^\circ$ and $\sin 72^\circ$. Thus the x and y coordinates of any vertex in the tiling can be described by four integers x_1, x_2, x_3 and x_4:

$$x = \frac{x_1 + x_2\sqrt{5}}{4}\,,$$
$$y = x_3\,\sin 36^\circ + x_4 \sin 72^\circ\,. \tag{2.2}$$

The integers x_1, x_2, x_3 and x_4 are the coordinates of the given vertex in a four-dimensional space. The set of points in $\mathbb{R}^4$ with integer coordinates is a lattice. Not all of the points of this lattice are used in our description of a Penrose tiling. Figure 3 shows the coordinates in $\mathbb{R}^4$ of ten points in the plane. These are the ten possible displacement vectors for a step, as one traces a path along the edges in a tiling of the plane. The four integers corresponding to such a vector give the changes in the coordinates in $\mathbb{R}^4$ as one moves along the edge of a tile. In the four-dimensional representation, a tile is specified by four points in $\mathbb{R}^4$; going around the tile along the edges involves taking steps in $\mathbb{R}^4$ given by the numbers in Fig. 3. Thus we see that a tile in $\mathbb{R}^4$ has displacement vectors along its edges that are independent of the location of the original tile in the plane. What does matter is the orientation of the original tile in the plane. Different orientations of the same tile in the plane correspond to tiles having different shapes in $\mathbb{R}^4$. In this sense, the number of types of tiles that occur in the four-dimensional representation is greater than the original value of two.

Given the four-dimensional representation of a tiling of the plane, the original tiling can be recovered using Eq. (2.2). This corresponds to projecting the four-dimensional tiling onto a two-dimensional subspace, with suitably defined coordinates in the subspace. Since objects in four dimensions are difficult to visualize, it is also of interest to consider projections onto three-dimensional subspaces. This can be done without creating self-intersections of the higher-dimensional tiling. Then a further projection results in the original tiling of the plane. These matters are discussed in the next section.

Rotating a Penrose tiling of the plane by 36° results in a tiling that has the same set of allowed directions for the edges, so the vertices can be described by four integers using the same procedure as for the original tiling. Using Eq. (2.2), one finds that the effect of a counterclockwise rotation by 36° on the four integers identifying a vertex can be described by multiplication by the following matrix:

$$T = \frac{1}{4}\begin{pmatrix} 1 & 5 & -10 & 0 \\ 1 & 1 & 2 & -4 \\ 1 & -1 & 0 & 2 \\ 0 & 2 & 2 & 2 \end{pmatrix}\,, \tag{2.3}$$

Because of the definition of T as a rotation in the plane by an angle of 36° (one tenth of a revolution), we have the following identities for higher powers of T:

$$
\begin{aligned}
T^5 &= -I\,, \\
T^{10} &= I\,. && (2.4)
\end{aligned}
$$
$$(2.5)$$

Here I denotes the 4×4 identity matrix. The matrix T is not orthogonal (that is, $T\,T^t$ is not the identity matrix), but it does possess the property

$$T\,M\,T\,M = I\,, \tag{2.6}$$

where the matrix M is defined to be

$$M = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.7}$$

The reason Eq. (2.6) is true is that the action of M is the same as a reflection about the y axis in the original two-dimensional space, and the conjugation of a rotation by a reflection in $\mathbb{R}^2$ is the inverse of the rotation.

The result of multiplying a column vector of four integers that represent a vertex in a Penrose tiling by the matrix T must again be a column vector of four integers, for reasons explained above. This statement yields some constraints on the possible coordinates in $\mathbb{R}^4$ for vertices in a Penrose tiling of the plane. For each row in the matrix T we have a constraint. For example, from the first row, we see that $x_1 + 5x_2 - 10x_3$ must be a multiple of four. Because the x_i are integers, this statement is equivalent to the statement that $x_1 + x_2 + 2x_3$ must be a multiple of four. Another way to prove this statement is to look at the coordinates of the fundamental displacement vectors shown in Fig. 3. For each of these steps, $x_1 + x_2 + 2x_3$ is a multiple of four, so $x_1 + x_2 + 2x_3$ must be a multiple of four for arbitrary combinations of these steps. Looking at the other rows in the matrix T we obtain further statements about sets of four integers that represent a vertex in a Penrose tiling. Some of this information is redundant. The information may be summarized as

$$
\begin{aligned}
x_1 + x_2 + 2x_3 &= 0 \ mod\ 4\,, \\
x_2 + x_3 + x_4 &= 0 \ mod\ 2\,.
\end{aligned} \tag{2.8}
$$

If we define new coordinates according to

$$
\begin{aligned}
x_1' &= \frac{x_1 + x_2 + 2x_3}{4}\,, \\
x_2' &= \frac{x_2 + x_3 + x_4}{2}\,, \\
x_3' &= x_3\,, \\
x_4' &= x_4\,,
\end{aligned} \tag{2.9}
$$

then we may use x_1', x_2', x_3' and x_4' as coordinates on $\mathbb{R}^4$. These coordinates have the property that all possible integer values are taken on when representing arbitrary finite sums of the fundamental displacement vectors. If complex numbers are used to describe

points in the plane, these sums become sums of unimodular complex numbers of the form $\exp(i\pi n_j/5)$, where the n_j are integers. The points obtained in this way represent vertices that would occur in tilings of the plane, if overlapping tiles were allowed. The fact that all possible integer values are taken on by the primed coordinates follows from the observation that (in the complex notation for points in the plane) $\exp(0)$ is represented by $(1,0,0,0)$, $\exp(2\pi i/5)+\exp(-2\pi i/5)$ is represented by $(0,1,0,0)$, $\exp(4\pi i/5)$ is represented by $(0,0,1,0)$, and $\exp(3\pi i/5)$ is represented by $(0,0,0,1)$.

With respect to the primed coordinates, the matrix T becomes

$$T' = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} . \tag{2.10}$$

The fact that only the numbers 1, -1 and 0 are present in this matrix indicates that an argument such as the one presented above for the matrix T will not give any constraints on the values of the new coordinates.

The x and y coordinates of any vertex in the tiling can be described by the four integers x'_1, x'_2, x'_3 and x'_4 as

$$\begin{aligned} x &= \frac{4x'_1 - 2x'_2 - x'_3 + x'_4 + (2x'_2 - x'_3 - x'_4)\sqrt{5}}{4} , \\ y &= x'_3 \sin 36^\circ + x'_4 \sin 72^\circ . \end{aligned} \tag{2.11}$$

Because of the complexity of this transformation, we prefer to use the unprimed coordinates, shown in Eq. (2.2).

III. THREE-DIMENSIONAL REPRESENTATIONS

In this section we describe three-dimensional representations of Penrose tilings of the plane. These have the advantage that they are easier to visualize than the four-dimensional representation described in the previous section. Two projections are described. We call these the "μ-projection" and the "(1,2)-projection." The μ-projection has the advantage that the original tiling of the plane can be obtained by a simple projection onto a certain two-dimensional subspace of $\mathbb{R}^3$. It has the disadvantage that only the x and y coordinates of the vertices are integers. The set of z values that occur are not integer multiples of a basic step. The (1,2)-projection has the advantage that the x, y and z coordinates of the vertices are all integers. The disadvantage is that the original tiling cannot be recovered by a further simple projection, although it can be reconstructed by a different means, as explained below.

A. The μ-projection

We define a mapping from $\mathbb{R}^4$ to $\mathbb{R}^3$ using the matrix

$$P^{(\mu)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mu \sin\frac{\pi}{5} & \mu \sin\frac{2\pi}{5} \end{pmatrix} , \tag{3.1}$$

where μ is a real number specified below. (The μ on the left-hand side of the equation is not an index.) Under this mapping, the vertices in $\mathbb{R}^4$ map to points in $\mathbb{R}^3$. The resulting arrangement of tiles in $\mathbb{R}^3$ can be projected onto the plane perpendicular to $(\sqrt{5}, -1, 0)$ to recover the original tiling. The proper choice for the value of μ results in a two-dimensional tiling in which the angles are the same as in the original tiling. We will use $\frac{1}{\sqrt{6}}(1, \sqrt{5}, 0)$ and $(0, 0, 1)$ as orthonormal basis vectors for the two-dimensional subspace perpendicular to $(\sqrt{5}, -1, 0)$, and we let x' and y' denote the coordinates of projected points with respect to this basis. The reason for this choice of subspace will become clear in the following calculation. The values of x' and y' for a point (x_1, x_2, x_3, x_4) in $\mathbb{R}^4$ can be obtained from

$$\begin{aligned} \begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{5}{\sqrt{6}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mu \sin\frac{\pi}{5} & \mu \sin\frac{2\pi}{5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{5}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & \mu \sin\frac{\pi}{5} & \mu \sin\frac{2\pi}{5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \\ &= \begin{pmatrix} 2\sqrt{\frac{2}{3}}\, x \\ \mu\, y \end{pmatrix} , \end{aligned} \tag{3.2}$$

where we have used Eq. (2.2). Thus, we must take

$$\mu = 2\sqrt{\frac{2}{3}} . \tag{3.3}$$

For arrangements of a finite number of Penrose tiles, the process of projecting onto the plane perpendicular to $(\sqrt{5}, -1, 0)$ can be implemented by viewing the three-dimensional representation from a viewpoint in the $(\sqrt{5}, -1, 0)$ direction. The image obtained in the limit as the distance from the tiles to the observer becomes infinite (together with suitable magnification of the image) is the projection onto the plane perpendicular to $(\sqrt{5}, -1, 0)$. Figure 5 shows the three-dimensional representation corresponding to the tiles in the plane shown in Fig. 4, using the μ-projection. Figure 6 contains a cut-out model of the object shown in Fig. 5. The lower portion is a frame to support the model. Its location in the fully assembled model corresponds roughly to the right half of the encompassing box in Fig. 5.

B. The (1,2)-projection

The (1,2)-projection is defined by the matrix

$$P^{(1,2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix} , \tag{3.4}$$

The numbers in the lowest row of this matrix provide the motivation for the name "(1,2)-projection." The numerical values in this matrix are close to those in $P^{(\mu)}$, so the resulting projected tilings are similar in appearance. However, since the coordinates x_3 and x_4 occur in the combination x_3+2x_4, it is not possible to obtain $y = x_3 \sin 36° + x_4 \sin 72°$ by simple algebraic operations. In spite of this, it is possible to recover the original tiling from the (1,2)-projection because the tiles in the projection indicate the types of tiles (narrow or wide) and their connections (shared edges) in the original tiling. Therefore, the information that is necessary to assemble the original tiling is contained in the (1,2)-projection. As mentioned above, this three-dimensional representation has the property that the x, y and z coordinates of the vertices are integers. Figure 2 shows the (1,2)-projection corresponding to Fig. 1.

IV. TWO-DIMENSIONAL REPRESENTATIONS

It is also possible to create a two-dimensional lattice representation of a Penrose tiling of the plane. One way to do this is to map the vertices in $\mathbb{R}^4$ to $\mathbb{R}^2$ using the matrix

$$\begin{pmatrix} \frac{1}{4} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 \end{pmatrix}, \tag{4.1}$$

The resulting tiling of the plane (see Fig. 7) resembles the original tiling, but the tiles are slightly different. Also, the set of directions of edges is no longer invariant under a rotation through an angle of 36°. There are three types of narrow tiles and three types of wide tiles (modulo reflections about the coordinate axes), corresponding to different orientations of the tiles in the original tiling. As above, the types of tiles (narrow or wide) and their connections to each other (shared edges) can be identified, so the original tiling can be reconstructed, but this process is not a simple projection operation. Although some of the vertices move quite a bit in this process, nearby vertices are moved by a similar amount, and no self-intersections of the pattern are created. This can be seen by applying the transformation to the points in Fig. 3. These displacement vectors occur along the edges of the tiles. The point (1,0) [which is represented by (4,0,0,0) in the four-dimensional representation] gets mapped to (1,0); the point $(\cos 36°, \sin 36°)$ [which is represented by (1,1,1,0) in the four-dimensional representation] gets mapped to $(\frac{3}{4}, \frac{1}{2})$, etc. Thus, the shapes of the tiles are not changed drastically, and the identities and arrangement of the original tiles can be read off of the transformed tiling. This is as in the case of the (1,2)-projection above. Another way to say this is that if a vertex is selected in Fig. 7, its original coordinates can be found by choosing an arbitrary path back to the origin along edges in the diagram. The steps in this path identify unit vectors which must be summed to obtain the coordinates of the vertex in Fig. 1.

The description of a tiling can be further simplified by removing the edges of the tiles from the list of information that is recorded about the tiling. It is sufficient to record only the locations of the vertices of the tiles on the two-dimensional lattice, as shown in Fig. 8. Only certain types of displacement vectors can occur as edges. These were mentioned in the preceding paragraph. Thus, given the information in Fig. 8, the edges can be reconstructed.

The description of a tiling using only a record of the locations of the vertices in the two-dimensional lattice representation, with no further information about the vertices (such

as which vertices belong to the same tile), is a description using a two-dimensional array of bits (binary digits). At each lattice point, a tiling vertex can either be present or absent. Not all arrays of bits represent tilings, however. Some displacements between vertices are forbidden. It must be possible to reconstruct a tiling from the bits, with all of the edges belonging to allowed tiles.

The two-dimensional representation provides a simple description of a tiling using integers. One advantage that it has is that it is "random access." It is possible to view the representation of an arbitrary patch of the tiling, using only information about that patch. Some other descriptions using integers do not have this property. For example, if a history of assembly is recorded (a sequence of tile types, edge numbers and orientations) then more data must be accessed to view a patch of the tiling. The two-dimensional representation is also useful for certain types of computer tiling programs and tile-overlap detection.

It should be noted that the two-dimensional lattice representation is not the same as simply superimposing a grid over the original tiling and moving the vertices to the nearest grid point. This would result in different shapes representing a given orientation of a tile, depending on how it was positioned relative to the grid. In the two-dimensional representation, a given orientation of a tile is always represented by the same shape, regardless of its position.

The two-dimensional representation also provides an interesting distance measure to work with when tiling the plane. This is the Euclidean metric in the lattice space. Because the original tiling is not obtained by a simple projection, this metric is difficult to describe in the original two-dimensional space. Figure 9 shows a tiling generated by a simple algorithm using the two-dimensional representation (basically placing new tiles as close as possible to the origin). The subjects of quasicrystallography and the spiral of Archimedes occur in the literature [12].

V. CONCLUSION

The description of Penrose tilings of the plane using lattices has a variety of applications, some of which are discussed in this paper and the references. As a further application, we describe in the appendix an algorithm for checking for tile overlap using calculations with the integers that occur in the four-dimensional representation. In this paper we have shown that lower-dimensional lattice descriptions of Penrose tilings of the plane are possible, but the reconstruction process becomes more complicated. Such representations may be of use in analyzing tilings, and they provide a simple means of describing a tiling.

An open question is the large-scale structure and curvature of the higher-dimensional representations, for tilings of the entire plane. For example, what would Fig. 2 look like as the number of tiles becomes very large? Another question is what the most efficient way to describe a tiling is. The most efficient method presented in this paper is the two-dimensional array of bits. Looking at Fig. 8, it is clear that some compression of this data is possible. Furthermore, if one of the bits is made unknown, it is still possible to reconstruct the tiling. Thus, further improvements in efficiency are possible.

APPENDIX A: CHECKING FOR TILE OVERLAP USING INTEGER MATH

The subject of this appendix is an algorithm for checking for tile overlap using calculations with integers. Additional information, such as vertex-vertex contact, is also given. We will consider here the case of a narrow tile with its long diagonal at an angle of 54° to the x-axis and a wide tile with its long diagonal at an angle of 36° to the x-axis. Other cases can be treated in a similar manner. Since the orientations of the tiles are fixed in this discussion, we need only introduce variables to describe the relative positioning of the tiles. We let the four integers x_1, x_2, x_3 and x_4 describe the displacement vector from the lower-left vertex of the narrow tile to the lower-left vertex of the wide tile. It is useful to think of the lower-left vertex of the narrow tile as being at the origin. Then the location of the wide tile is determined by x_1, x_2, x_3 and x_4. As we move the wide tile around the plane (without rotating it), some of the positions will have contact between the tiles; others will have no contact. The set of positions for the lower-left vertex of the wide tile for which there is contact is a hexagon. Opposite sides of the hexagon are parallel, but the hexagon is not regular. The second set of Mathematica instructions below draws a diagram to go along with this discussion. For each edge of the hexagon, we define a line by extending the edge infinitely in both directions. The side of the line on which a given test point is located is determined by the sign of the dot product of a normal vector to the line and the vector difference between the test point and a point on the line. We choose the normal vector to point to the inside of the hexagon. Because the normal vector is at 90° to the line, $\sin 36^\circ$ appears in its x-component, as may be seen from Eq. (2.2) and the fact that $\sin 72^\circ = \sin 36^\circ(1+\sqrt{5})/2$. A simplification that therefore occurs is that an overall factor of $\sin 36^\circ$ may be dropped in computing the above-mentioned sign. The problem reduces to finding the sign of numbers of the form $a + b\sqrt{5}$, which can be done using integer math (see `sign[]`, below). A test point strictly inside the hexagon will have all six signs positive. If at least one sign is negative, the point is strictly outside the hexagon. If exactly one of the signs is zero (and the others positive), the point is on one of the edges of the hexagon, and it is not one of the endpoints. This means we have vertex-edge or partial edge contact between the tiles. If two of the signs are zero (and the others positive), the test point is at one of the vertices of the hexagon. This means the tiles have vertex-vertex contact. Two of the vertices of the hexagon represent perfect edge contact. These are tested for at the beginning of the program `check[]`.

The program `check[]` calculates the six signs and returns a character string describing the overlap. The second set of Mathematica instructions below draws a figure showing how the plane is divided up. The narrow tile with its lower-left vertex at the origin is shown. The wide tile is located at an arbitrary position. The dots summarize the information obtained from `check[]` at some representative points. These include special points, such as the two points for which we have perfect edge contact (large filled circles), points which indicate vertex-vertex contact (medium filled circles), and vertex-edge or partial edge contact (medium unfilled circles). Most of the points in the figure are small circles. The unfilled ones are points that indicate overlap with nonzero area, and the filled ones are points that indicate no overlap at all.

```
(* ========= check for overlapping tiles, using integer math:  ========= *)

(* the sign of a + b Sqrt[5] : *)
sign[a_, b_] := Switch[Sign[a] Sign[b], 1, Sign[a], 0,
     If[a == 0, Sign[b], Sign[a]], -1, Sign[a] Sign[a^2 - 5 b^2]]

(* some parameters: *)
Evaluate[Table[b[i, j], {i, 3},{j, -1, 1, 2}]] =
     {{{-1, -1}, {3, 1}}, {{-4, 0}, {8, 0}}, {{-8, 0}, {4, 4}}}

(* check for overlap: *)
check[x1_, x2_, x3_, x4_] := (
     If[{x1, x2, x3, x4} == {1, 1, 1, 0} || {x1, x2, x3, x4} == {-4,0,0,0},
     Return["perfect edge contact"]];
     p[1] = {2 x3 + x4, x4};
     p[2] = {-x1 + x3 + 3 x4, -x2 + x3 + x4};
     p[3] = {-x1 - 5 x2 - 2 x3 + 4 x4, -x1 - x2 + 2 x3}; signs =
     Sort@Flatten@Table[-j Apply[sign, p[i] - b[i,j]], {i, 3}, {j,-1,1,2}];
     If[signs[[1]] == -1, Return["no contact"]];
     If[signs[[2]] ==  0, Return["vertex-vertex contact"]];
     If[signs[[1]] ==  0, Return["vertex-edge or partial edge contact"]];
     "nonzero-area overlap" )

(* ============= The instructions below draw the figure. ============= *)

f["perfect edge contact"]                    = {Disk, .1}
f["no contact"]                              = {Disk, .025}
f["vertex-vertex contact"]                   = {Disk, .05}
f["vertex-edge or partial edge contact"]  = {Circle, .05}
f["nonzero-area overlap"]                    = {Circle, .025}

v1 = {c1, s1} = {Cos[Pi/5], Sin[Pi/5]}
v2 = {c2, s2} = {Cos[2 Pi/5], Sin[2 Pi/5]}

DisksCircles = {};  Do[x = N[(x1 + x2 Sqrt[5])/4]; y = N[x3 s1 + x4 s2];
     If[Abs[x] < 2 && Abs[y] < 2, temp = f[check[x1, x2, x3, x4]];
     AppendTo[DisksCircles, temp[[1]][{x,y}, temp[[2]]]]],
     {x1, -5, 5}, {x2, -3, 3}, {x3, -2, 2}, {x4, -2, 2}]

Show[Graphics[{DisksCircles, Line[{{0, 0}, v1, v1 + v2, v2, {0, 0}}],
     Line[Map[# + {c1 - 1/2, -s2 - s1}&,
     {{0, 0}, {1, 0}, {1, 0} + v2, v2, {0, 0}}]]}],
     AspectRatio -> Automatic, Axes -> True]
```

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FIGURE CAPTIONS

Figure 1:

An arrangement of Penrose tiles in the plane. The vertices of the tiles do not lie on a lattice.

Figure 2:

A three-dimensional representation of the tiling shown in Fig. 1. The x, y and z coordinates of the vertices are integers.

Figure 3:

The fundamental displacement vectors in the plane, and their coordinates in $\mathbb{R}^4$

Figure 4:

The arrangement of Penrose tiles referred to in Figs. 5 and 6.

Figure 5:

The μ-projection of the tiling shown in Fig. 4.

Figure 6:

A three-dimensional cut-out model of the representation shown in Fig. 5. The letters indicate points that come together to a single point in the assembled model. For example, the four points labeled “A” come together in the final assembly. When viewed from a distance in the direction of the arrow, the model looks like the image shown in Fig. 4. Illuminating the model from the left helps to eliminate distracting shadows. Further information about the assembly of the model is given in the text.

Figure 7:

A two-dimensional lattice representation of the tiling shown in Fig. 1.

Figure 8:

A representation of the tiling shown in Fig. 1 as a two-dimensional array of bits. The original tiling can be reconstructed from this information alone.

Figure 9:

An arrangement of Penrose tiles generated by a simple algorithm using the two-dimensional lattice representation.

Figure 1

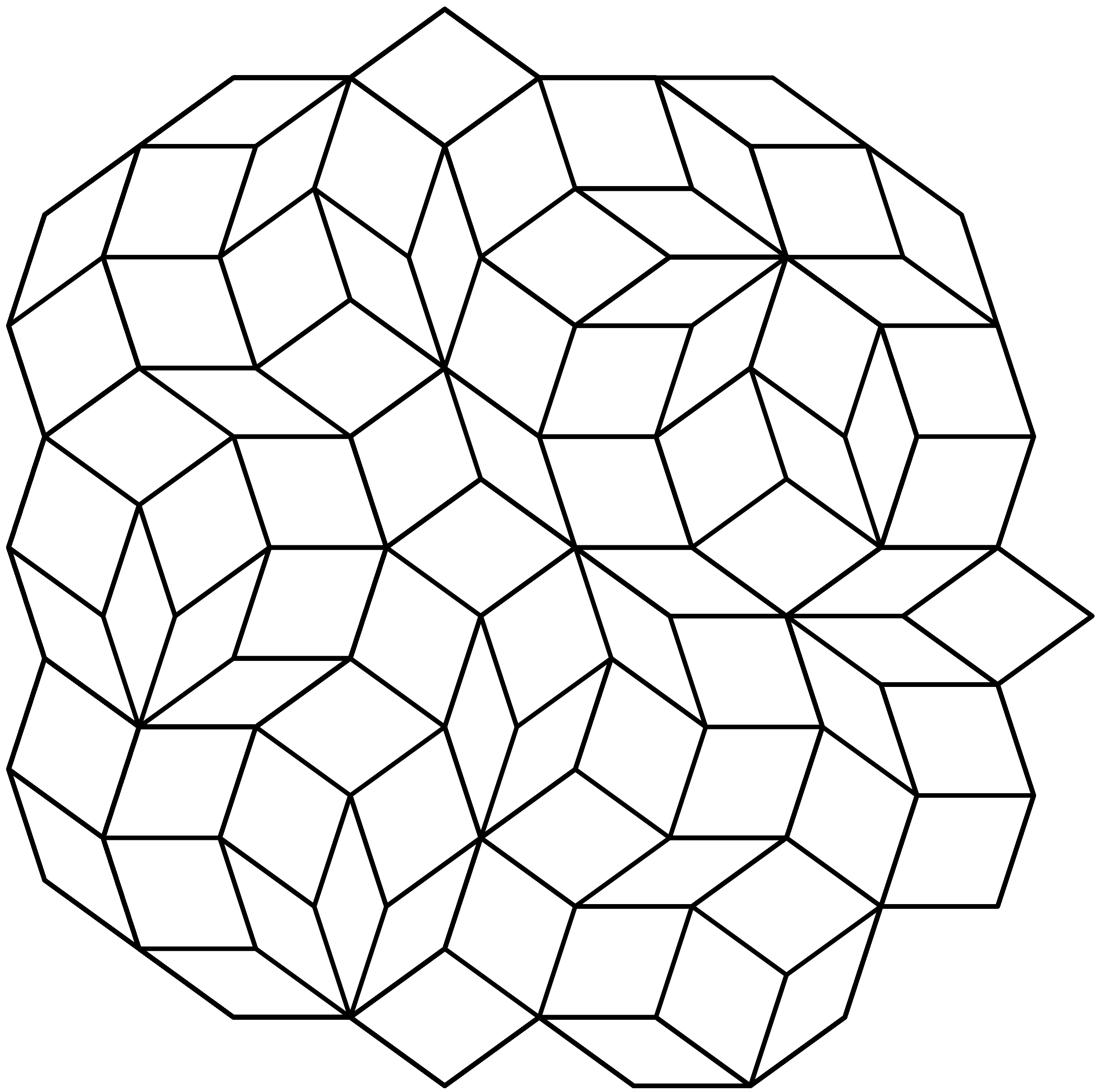

Figure 2

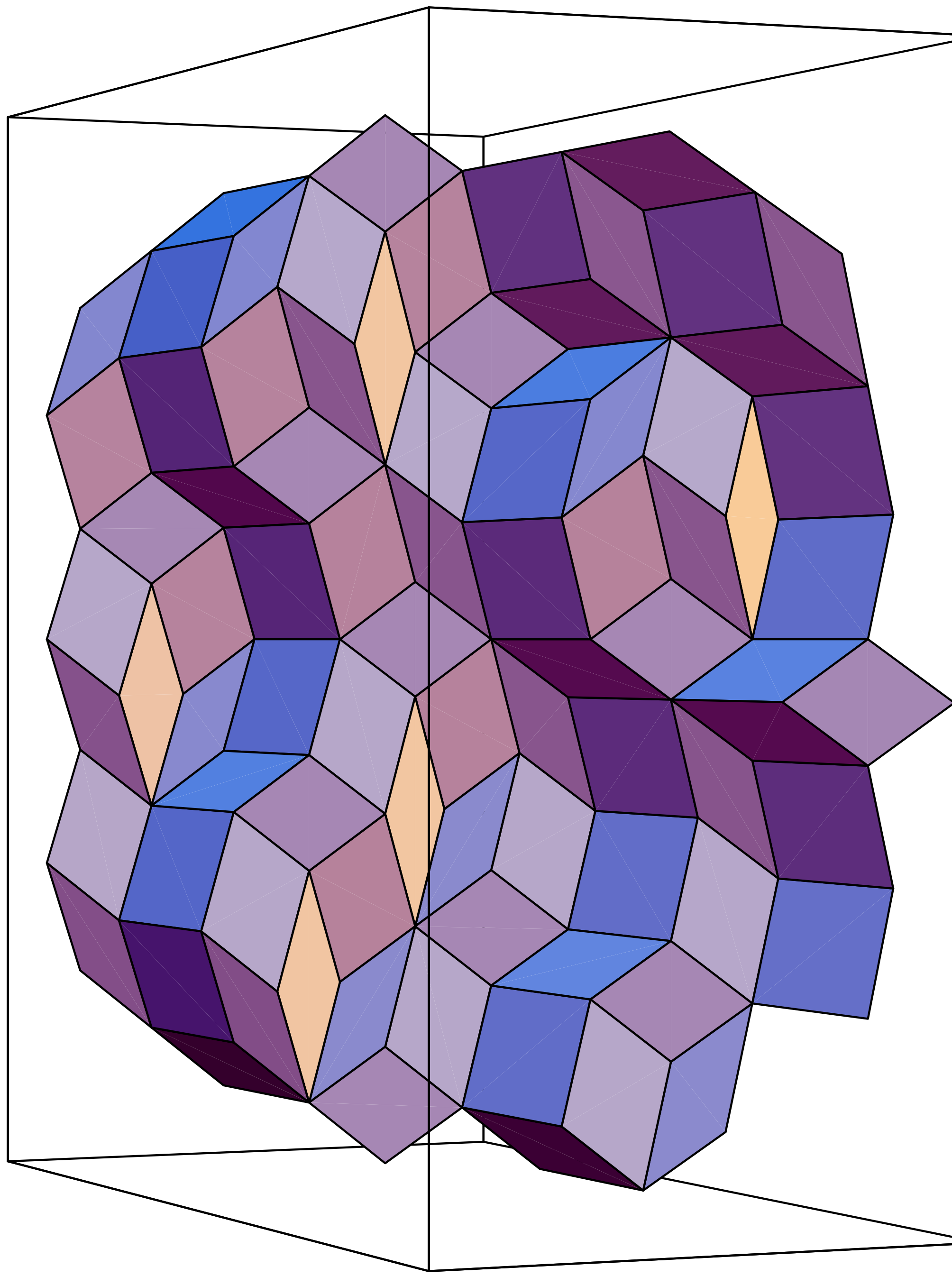

Figure 3

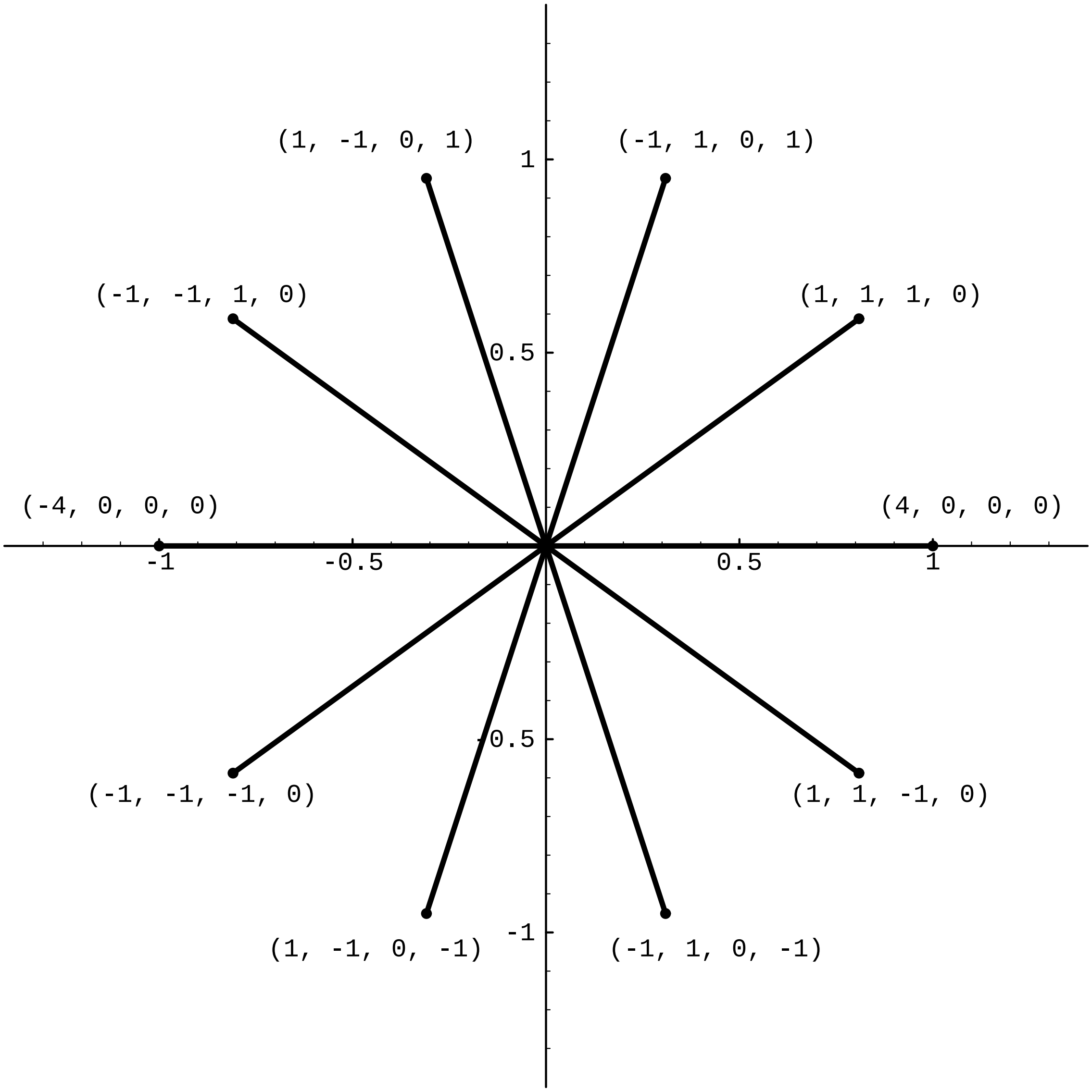
(1, -1, 0, 1)
(-1, 1, 0, 1)
(-1, -1, 1, 0)
(1, 1, 1, 0)
(-4, 0, 0, 0)
(4, 0, 0, 0)
(-1, -1, -1, 0)
(1, 1, -1, 0)
(1, -1, 0, -1)
(-1, 1, 0, -1)
1
0.5
-0.5
-1
-1
-0.5
0.5
1

Figure 4

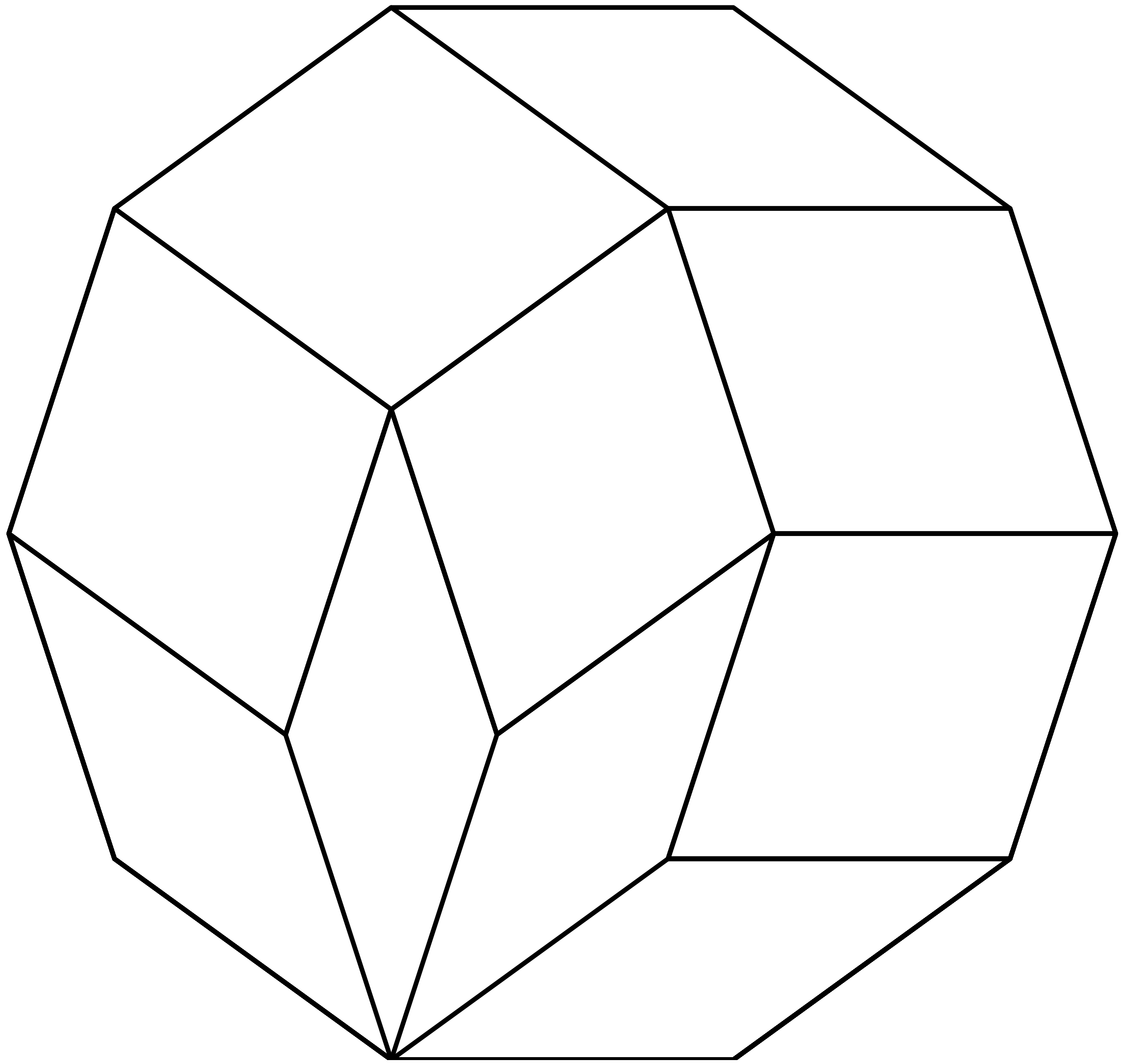

Figure 5

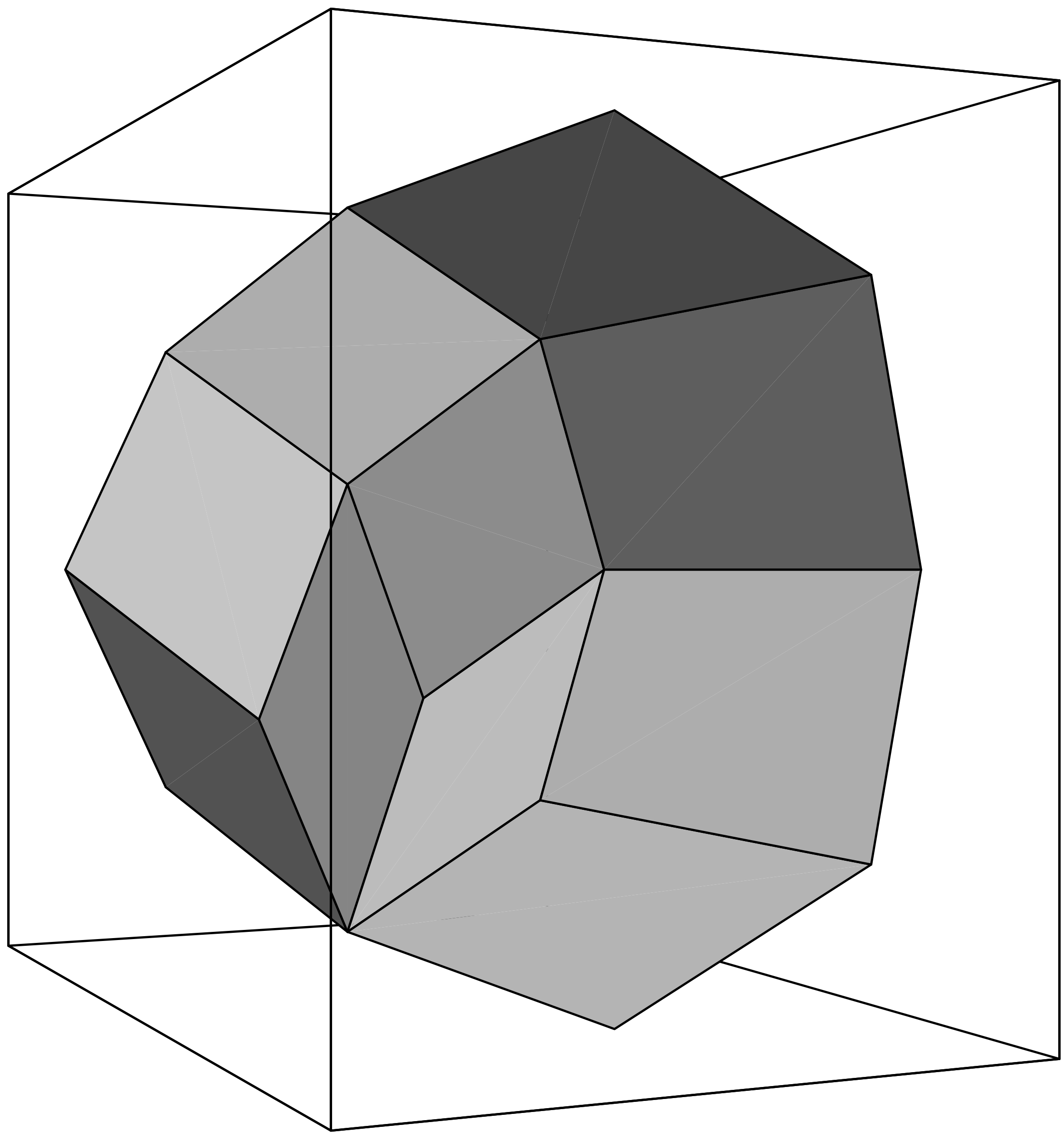

Figure 6

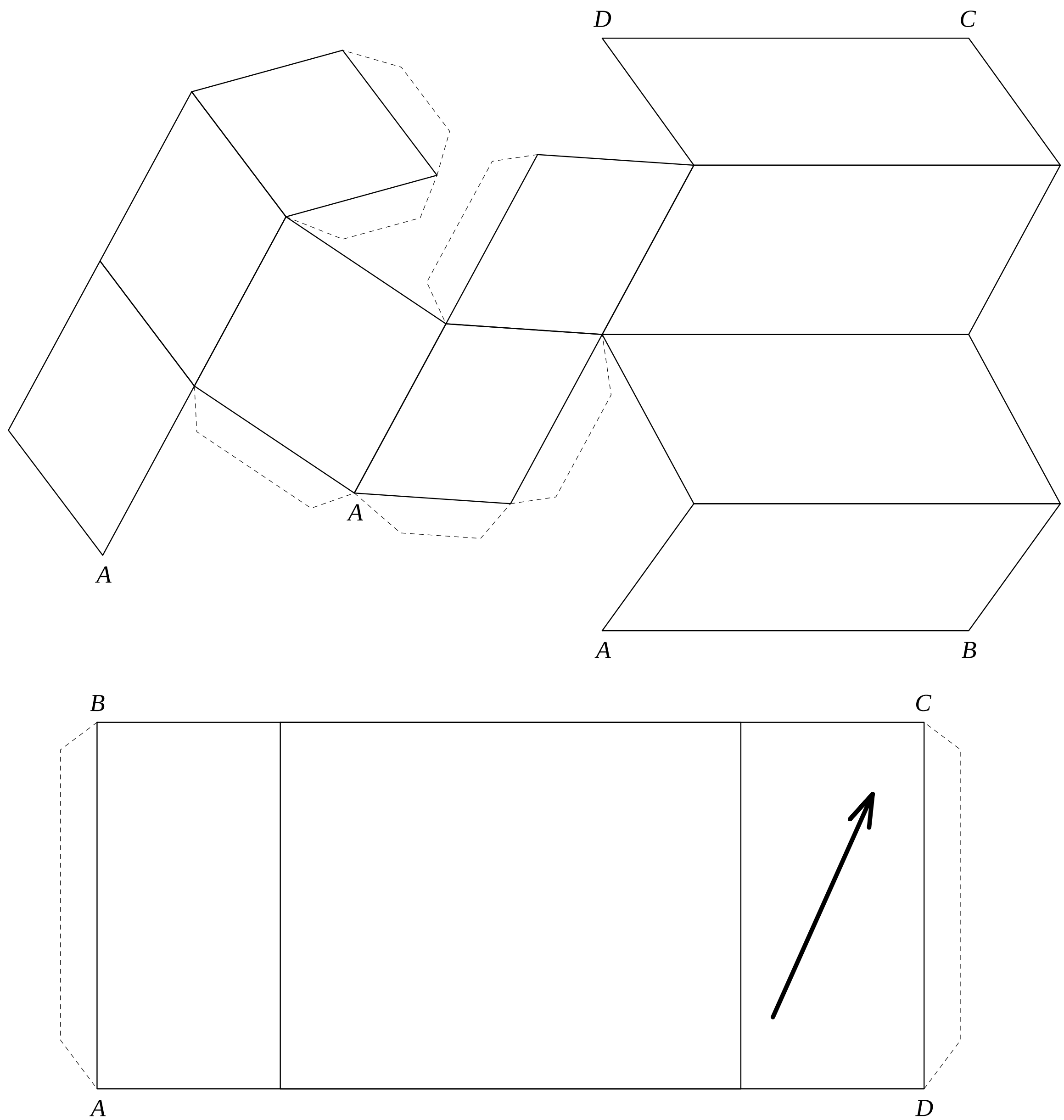

Figure 7

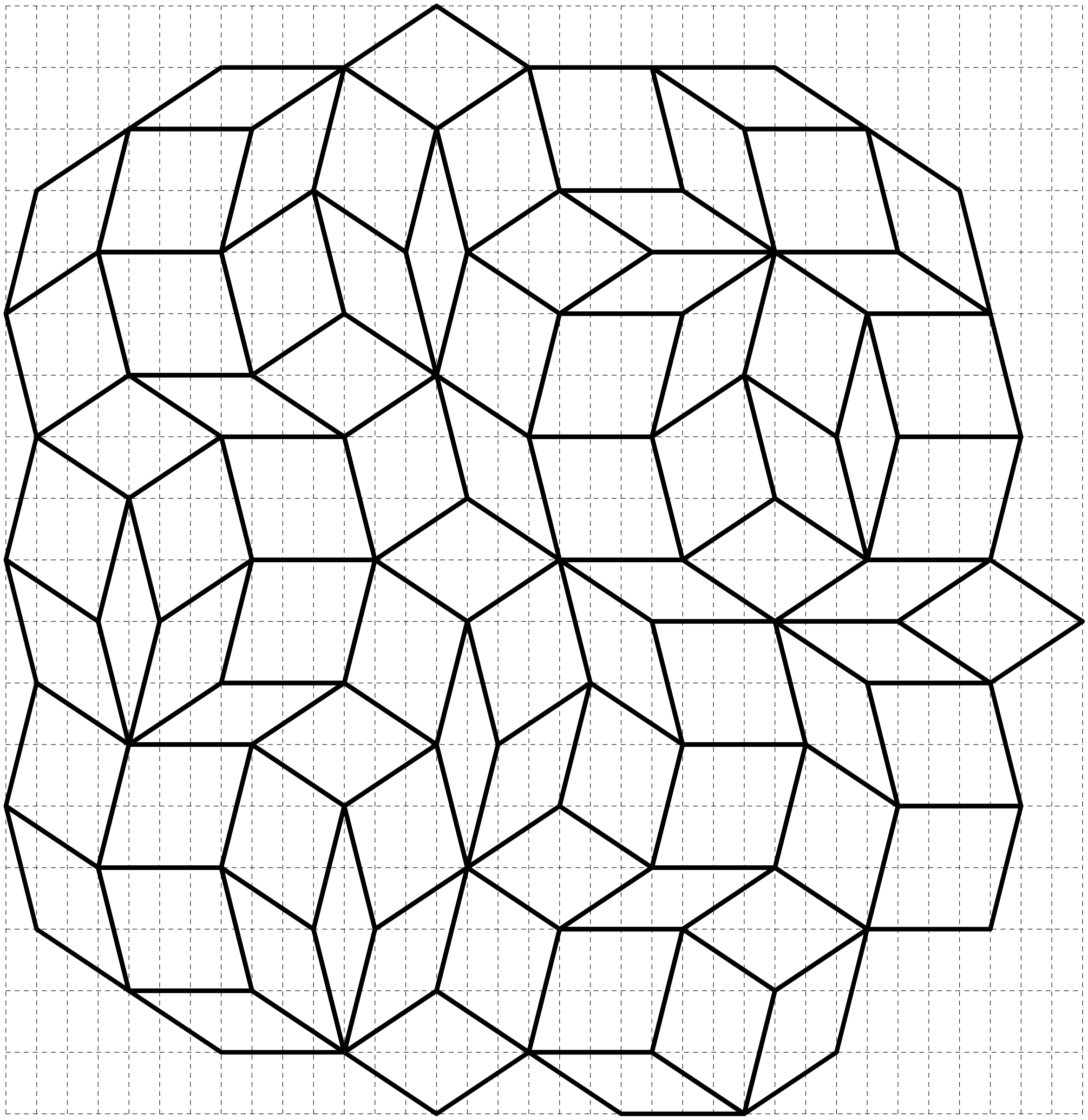

Figure 8

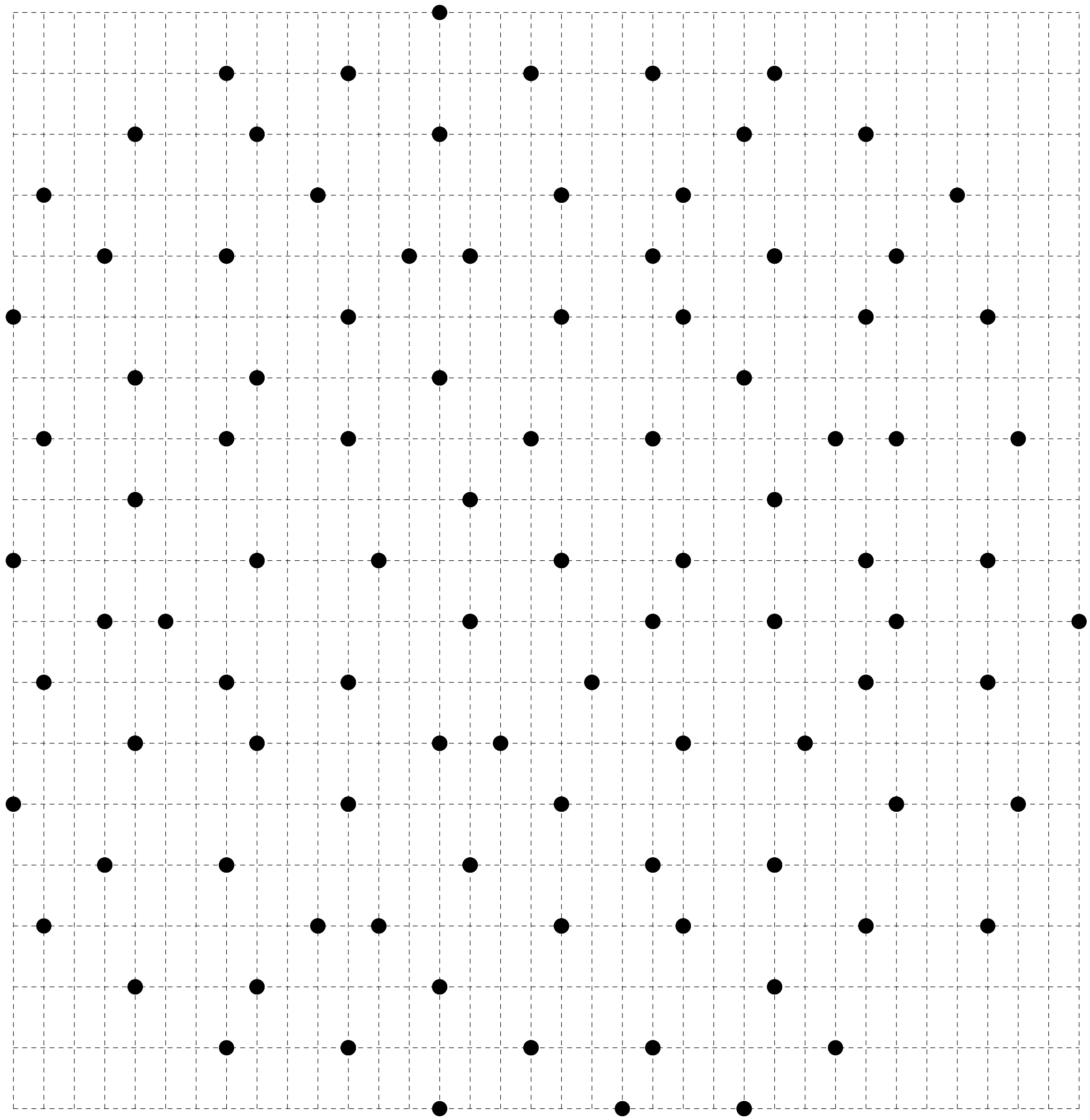

Figure 9